THERMAL CONDUCTIVITY OF A HOLLOW CYLINDER WITH A BOUNDARY CONDITION OF THE FOURTH KIND AT THE OUTER SURFACE

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The temperature field of an infinite hollow cylinder is examined for various boundary conditions at the inner surface and a boundary condition of the fourth kind at the outer surface.

We consider the problem of nonsteady heat conduction for the case of an infinite hollow cylinder introduced into an unbounded medium. The transfer of heat between the cylinder and the medium obeys the Fourier law (boundary condition of the fourth kind). The same problem was solved for a solid cylinder in [1]. In the case of a hollow cylinder, it is desirable to consider various boundary conditions at the inner surface.

The problem is formulated as follows:

$$\frac{\partial \theta_1}{\partial \tau} = a_1\left(\frac{\partial^2 \theta_1}{\partial r^2} + \frac{1}{r}\frac{\partial \theta_1}{\partial r}\right), \quad \tau > 0;\ r_0 < r < r_1, \tag{1}$$

$$\frac{\partial \theta_2}{\partial \tau} = a_2\left(\frac{\partial^2 \theta_2}{\partial r^2} + \frac{1}{r}\frac{\partial \theta_2}{\partial r}\right), \quad \tau > 0;\ r_1 < r < \infty, \tag{2}$$

$$\theta_1(r,\ 0) = \theta_2(r,\ 0) = 0, \tag{3}$$

$$\theta_2(\infty,\ \tau) = 0, \tag{4}$$

$$\theta_1(r_1,\ \tau) = \theta_2(r_1,\ \tau), \tag{5}$$

$$\lambda_1 \frac{\partial \theta_1(r_1,\ \tau)}{\partial r} = \lambda_2 \frac{\partial \theta_2(r_1,\ \tau)}{\partial r}. \tag{6}$$

At the inner surface of the cylinder it is best to consider the general case of a boundary condition of the third kind from which other boundary conditions can be obtained:

$$-A\frac{\partial \theta_1(r_0,\ \tau)}{\partial r} + B\theta_1(r_0,\ \tau) = A\frac{q}{\lambda_1} + B\theta_c. \tag{7}$$

Problem (1)–(7) is solved by means of a Laplace transform with respect to the variable τ. In terms of transforms, the solutions are written as

$$T_1 = \left(A\frac{q}{\lambda_1 s} + B\frac{\theta_c}{s}\right)\times$$

$$\times\left[\frac{K_0(\Gamma_1 r_1)K_1(\Gamma_2 r_1) - k_\varepsilon K_0(\Gamma_2 r_1)K_1(\Gamma_1 r_1)}{k_\varepsilon K_0(\Gamma_2 r_1)V_1(\Gamma_1 r_1) + K_1(\Gamma_2 r_1)V_0(\Gamma_1 r_1)} I_0(\Gamma_1 r) - \right.$$

$$\left. - \frac{k_\varepsilon K_0(\Gamma_2 r_1)I_1(\Gamma_1 r_1) + I_0(\Gamma_1 r_1)K_1(\Gamma_2 r_1)}{k_\varepsilon K_0(\Gamma_2 r_1)V_1(\Gamma_1 r_1) + K_1(\Gamma_2 r_1)V_0(\Gamma_1 r_1)} K_0(\Gamma_1 r)\right], \tag{8}$$

$$T_2 = -\left(A\frac{q}{\lambda_1 s} + B\frac{\theta_c}{s}\right)k_\varepsilon \times$$

$$\times\frac{K_1(\Gamma_1 r_1)I_0(\Gamma_1 r_1) + K_0(\Gamma_1 r_1)I_1(\Gamma_1 r_1)}{k_\varepsilon K_0(\Gamma_2 r_1)V_1(\Gamma_1 r_1) + K_1(\Gamma_2 r_1)V_0(\Gamma_1 r_1)} K_0(\Gamma_2 r). \tag{9}$$

Here

$$\Gamma_1 = \sqrt{\frac{s}{a_1}}, \qquad \Gamma_2 = \sqrt{\frac{s}{a_2}},$$

$$V_1(\Gamma_1 r_1) = R_1(\Gamma_1 r_0)I_1(\Gamma_1 r_1) - R_0(\Gamma_1 r_0)K_1(\Gamma_1 r_1), \tag{10}$$

$$V_0(\Gamma_1 r_1) = R_0(\Gamma_1 r_0)K_0(\Gamma_1 r_1) + R_1(\Gamma_1 r_0)I_0(\Gamma_1 r_1), \tag{11}$$

$$R_0(\Gamma_1 r_0) = A\Gamma_1 I_1(\Gamma_1 r_0) + BI_0(\Gamma_1 r_0), \tag{12}$$

$$R_1(\Gamma_1 r_0) = A\Gamma_1 K_1(\Gamma_1 r_0) - BK_0(\Gamma_1 r_0). \tag{13}$$

We use the Laplace inversion formula and go over from the integral in the domain of the complex variable to the integral of the real variable. To determine the residues at the point $s = 0$, we use the asymptotic expansion formulas for modified Bessel functions.

The general form of the solution is written as

$$\theta_1 = p_1 - \frac{8k_\varepsilon\left(A\frac{q}{\lambda_1} + B\theta_c\right)r_0^2}{\pi^3 a_0 r_1^2}\times$$

$$\times\int_0^\infty\left[A\frac{\mu}{r_0}B_1\left(\mu\frac{r}{r_0}\right) + BB_0\left(\mu\frac{r}{r_0}\right)\right]\times$$

$$\times\exp\left(-\frac{\mu^2 a_1\tau}{r_0^2}\right)d\mu\times$$

$$\times\left\{\mu^3\left[\psi^2(\mu) + \varphi^2(\mu)\right]\right\}^{-1} \tag{14}$$

$$\theta_2 = p_2 - \frac{4k_\varepsilon\left(A\frac{q}{\lambda_1} + B\theta_c\right)r_0}{\pi^2 r_1}\times$$

$$\times\int_0^\infty\frac{\Phi\left(\mu a_0\frac{r}{r_0}\right)\exp\left(-\mu^2 a_1\frac{\tau}{r_0^2}\right)d\mu}{\mu^2\left[\psi^2(\mu) + \varphi^2(\mu)\right]}. \tag{15}$$

Here

$$B_0\left(\mu\frac{r}{r_0}\right) = J_0\left(\mu\frac{r}{r_0}\right)Y_0(\mu) - Y_0\left(\mu\frac{r}{r_0}\right)J_0(\mu), \tag{16}$$

$$B_1\left(\mu\frac{r}{r_0}\right) = J_0\left(\mu\frac{r}{r_0}\right)Y_1(\mu) - Y_0\left(\mu\frac{r}{r_0}\right)J_1(\mu), \tag{17}$$

$$\psi(\mu) = k_\varepsilon J_0\left(\mu a_0\frac{r_1}{r_0}\right)\left[BB_{01}(\mu) + A\frac{\mu}{r_0}B_{11}(\mu)\right] -$$

$$- J_1\left(\mu a_0\frac{r_1}{r_0}\right)\left[BB_{00}(\mu) + A\frac{\mu}{r_0}B_{10}(\mu)\right], \tag{18}$$

$$\varphi(\mu) = k_\varepsilon Y_0\left(\mu a_0 \frac{r_1}{r_0}\right)\left[BB_{01}(\mu) + A\frac{\mu}{r_0}B_{11}(\mu)\right] - $$
$$- Y_1\left(\mu a_0 \frac{r_1}{r_0}\right)\left[BB_{00}(\mu) + A\frac{\mu}{r_0}B_{10}(\mu)\right], \quad (19)$$

$$\Phi\left(\mu a_0 \frac{r}{r_0}\right) =$$
$$= k_\varepsilon M_0\left(\mu a_0 \frac{r}{r_0}\right)\left[BB_{01}(\mu) + A\frac{\mu}{r_0}B_{11}(\mu)\right] -$$
$$- M_1\left(\mu a_0 \frac{r}{r_0}\right)\left[BB_{00}(\mu) + A\frac{\mu}{r_0}B_{10}(\mu)\right], \quad (20)$$

$$B_{00}(\mu) = J_0(\mu) Y_0\left(\mu \frac{r_1}{r_0}\right) - Y_0(\mu) J_0\left(\mu \frac{r_1}{r_0}\right), \quad (21)$$

$$B_{01}(\mu) = J_0(\mu) Y_1\left(\mu \frac{r_1}{r_0}\right) - Y_0(\mu) J_1\left(\mu \frac{r_1}{r_0}\right), \quad (22)$$

$$B_{10}(\mu) = J_1(\mu) Y_0\left(\mu \frac{r_1}{r_0}\right) - Y_1(\mu) J_0\left(\mu \frac{r_1}{r_0}\right), \quad (23)$$

$$B_{11}(\mu) = J_1(\mu) Y_1\left(\mu \frac{r_1}{r_0}\right) - Y_1(\mu) J_1\left(\mu \frac{r_1}{r_0}\right), \quad (24)$$

$$M_0\left(\mu a_0 \frac{r}{r_0}\right) = J_0\left(\mu a_0 \frac{r_1}{r_0}\right) Y_0\left(\mu a_0 \frac{r}{r_0}\right) -$$
$$- Y_0\left(\mu a_0 \frac{r_1}{r_0}\right) J_0\left(\mu a_0 \frac{r}{r_0}\right), \quad (25)$$

$$M_1\left(\mu a_0 \frac{r}{r_0}\right) = J_1\left(\mu a_0 \frac{r_1}{r_0}\right) Y_0\left(\mu a_0 \frac{r}{r_0}\right) -$$
$$- Y_1\left(\mu a_0 \frac{r_1}{r_0}\right) J_0\left(\mu a_0 \frac{r}{r_0}\right). \quad (26)$$

As $B \to \infty$, we obtain the boundary condition of the first kind,

$$\theta_1(r_0, \tau) = \theta_c. \quad (27)$$

In this case $p_1 = p_2 = \theta_c$; in Eqs. (14)–(20), it is necessary to set $A = 0$; $B = 1$.

As $A \to \infty$, we obtain the boundary condition of the second kind

$$-\frac{\partial \theta_1(r_0, \tau)}{\partial r} = \frac{q}{\lambda_1}. \quad (28)$$

In this case,

$$p_1 = \frac{q r_0}{\lambda_1 \sqrt{a_2}} \times$$
$$\times\left[k_0\sqrt{a_1}\left(\ln\frac{2\sqrt{a_2\tau}}{r_1} - \frac{1}{2}C\right) + \sqrt{a_2}\ln\frac{r}{r_1}\right], \quad (29)$$

$$p_2 = \frac{q k_\varepsilon a_0 r_0}{\lambda_1}\left(\ln\frac{2\sqrt{a_2\tau}}{r} - \frac{1}{2}C\right), \quad (30)$$

$C = 0.577216$ is the Euler constant. In Eqs. (14)–(20), we must now set $A = 1$; $B = 0$. For the usual boundary condition of the third kind, $q = 0$; $A = \lambda_1$; $B = \alpha$,

$$-\frac{\partial \theta_1(r_0, \tau)}{\partial r} = \frac{\alpha}{\lambda_1}[\theta_c - \theta_1(r_0, \tau)]. \quad (31)$$

In this case, $p_1 = p_2 = \theta_c$.

If the thermophysical properties of the hollow cylinder and the medium are the same $k_\varepsilon = a_0 = 1$. Then the equations for θ_1 and θ_2 degenerate into a single equation for a homogeneous hollow cylinder of infinite radius. In this case, the solutions are written as follows:

for the boundary condition of the first kind (27),

$$\frac{\theta_c - \theta_1}{\theta_c} = \frac{2}{\pi}\int_0^\infty \frac{B_0\left(\mu \frac{r}{r_0}\right)\exp(-\mu^2 a_1 \tau / r_0^2) d\mu}{\mu[J_0^2(\mu) + Y_0^2(\mu)]}, \quad (32)$$

for the boundary condition of the second kind (28),

$$\theta_1 = \frac{2 q r_0}{\lambda_1}\left\{\ln\frac{2\sqrt{a_1\tau}}{r} - \frac{1}{2}C - \right.$$
$$\left. - \frac{1}{\pi}\int_0^\infty \frac{B_1\left(\mu \frac{r}{r_0}\right)\exp(-\mu^2 a_1 \tau / r_0^2) d\mu}{\mu^2[J_1^2(\mu) + Y_1^2(\mu)]}\right\}, \quad (33)$$

for the boundary condition of the third kind (31),

$$\frac{\theta_c - \theta_1}{\theta_c} = \frac{2\mathrm{Bi}}{\pi} \times$$
$$\times \int_0^\infty \frac{\left[\mathrm{Bi}\, B_0\left(\mu \frac{r}{r_0}\right) + \mu B_1\left(\mu \frac{r}{r_0}\right)\right]\exp(-\mu^2 a_1 \tau / r_0^2) d\mu}{\mu\{[J_0(\mu)\mathrm{Bi} + \mu J_1(\mu)]^2 + [Y_0(\mu)\mathrm{Bi} + \mu Y_1(\mu)]^2\}}. \quad (34)$$

The improper integrals in the solutions obtained converge rapidly, thanks to the presence of an exponential factor.

The required temperature profiles can be obtained on a computer by calculating the temperatures of the hollow cylinder and the external medium from Eqs. (14) and (15). The solutions obtained can also be used to calculate the temperature stresses in a hollow cylinder in perfect thermal contact with the medium at the outer surface.

NOTATION

t_1, t_2, and t_0 are the temperatures of the cylinder and external medium and the initial temperature, respectively; t_c is the temperature at the inner surface of the cylinder for a boundary condition of the first kind, or the temperature of the medium inside the cylinder for a boundary condition of the third kind; $\theta_1 = t_1 - t_0$; $\theta_2 = t_2 - t_0$; $\theta_c = t_c - t_0$; a_1 and a_2 are the thermal diffusivities of the cylinder and external medium, respectively; λ_1 and λ_2 are the thermal conductivities of the cylinder and the external medium, respectively; r_0 and r_1 are the inside and outside radii of the cylinder, respectively; τ is the time; q is the heat flux at the inner surface of the cylinder; α is the coefficient of heat transfer to the inner surface of the cylinder for a boundary condition of the third kind; s is the Laplace transform parameter; T_1 and T_2

are, respectively, the inverse Laplace transforms of the functions θ_1 and θ_2; $k_\varepsilon = (\lambda_1/\lambda_2)(a_2/a_1)^{1/2}$; $a_0 = (a_1/a_2)^{1/2}$; I_0, K_0, I_1, K_1 are modified Bessel functions of the first and second kinds and of zero and first order, respectively; $Bi = \alpha r_0/\lambda_1$ is the Biot number; $\Gamma_1 = (s/a_1)^{1/2}$; $\Gamma_2 = (s/a_2)^{1/2}$; $\mu = \Gamma_1 r_0/i$; J_0, Y_0, J_1, Y_1 are Bessel functions of the first and second kind of zero and first order, respectively.

REFERENCE

1. A. V. Luikov, Theory of Heat Conduction [in Russian], GTTI, 1952.

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